§4: Using the derivative

Plan: We will cover the following in the remaining time we have:

§4.1 to 4.3

§5.1 to 5.5

§ 6.2, 6.3, 6.6 and maybe 6.7

Some will be streamlined faster than others.

§4.1: Local maxima and minima

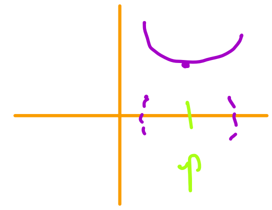
Obj: We use principles from Chp 2 w/ the tools developed in Chp 3

Rmk: Recall the following for a function f and interval I in the domain of f.

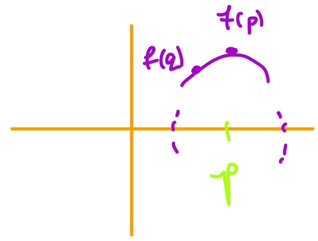
- $f' > 0$ on $I \implies f \nearrow$ on I
- $f' < 0$ on $I \implies f \searrow$ on I
- $f'' > 0$ on $I \implies f$ concave up on I
- $f'' < 0$ on $I \implies f$ concave down on I

Def: Let f be a function and $p \in \mathrm{dom}(f)$.

- f has a "local minimum" at p if for all $q \in \mathrm{dom}(f)$ near p, one has $f(p) \leq f(q)$

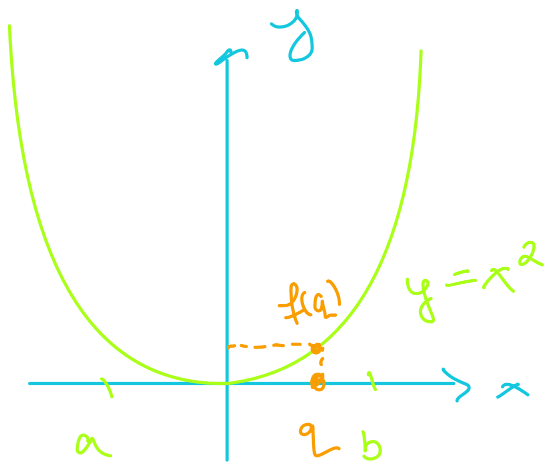


- f has a "local maximum" at p if for all $q \in \mathrm{dom}(f)$ near p, one has $f(p) \geq f(q)$.



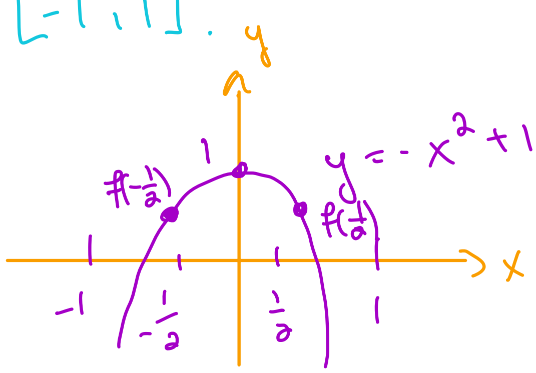
Ex: $f(x) = x^2$

- For all $q \in (a,b)$, one has $f(q) \geq f(0) = 0$.



- Thus, 0 is a local min of $f(x) = x^2$

Ex: Find a local maximum of $f(x) = -x^2+1$ on $[-1,1]$.



For all $p \in [-1,1]$, we have $f(p) \leq f(0) = 1$

Hence, 0 is local max.

Def: Let f be a function. A pt. $p \in \mathrm{dom}(f)$ is called a "critical pt." if $f'(p) = 0$ or $f'(p)$ is undefined, and for any such pt, we call $f(p)$ a "critical value".

Ex:

- $f(x) = \frac{1}{x}$ at $x = 0$

  $f'(x) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-2} = -\frac{1}{x^2}$, $f'(0)$ is undefined

  $\Rightarrow 0$ is a critical pt.

- $f(x) = \ln(x)$ at $x = 0$

  $f'(x) = \frac{1}{x} \Rightarrow f'(0) = \text{undefined} \Rightarrow 0$ is a critical pt

- $f(x) = \frac{1}{\ln(x)}$ at $x = 1$

  $f'(x) = \frac{d}{dx}\left((\ln(x))^{-1}\right) = -1 \cdot (\ln(x))^{-2} \cdot \frac{1}{x} = -\frac{1}{x \ln(x)^2}$

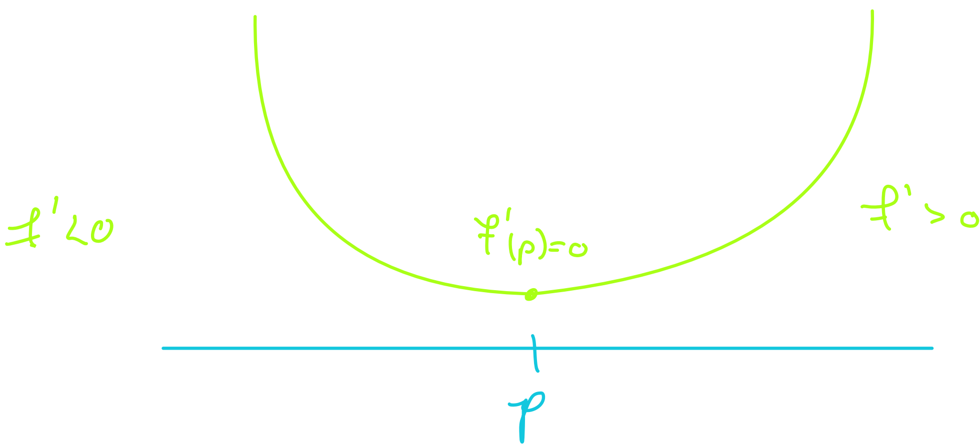
$\Rightarrow f'(1) = -\frac{1}{1 \cdot \underbrace{\ln(1)}_{0}{}^2} = \text{undef.} \Rightarrow 1$ is a crit. pt.

Theorem: (1st derivative test for extrema) Suppose f is a function whose derivative exists. Let $p \in \mathrm{dom}(f)$ be a critical pt. of f.

- If f' changes from neg. to pos. at p as we move left to right, then f has a local min. at p.

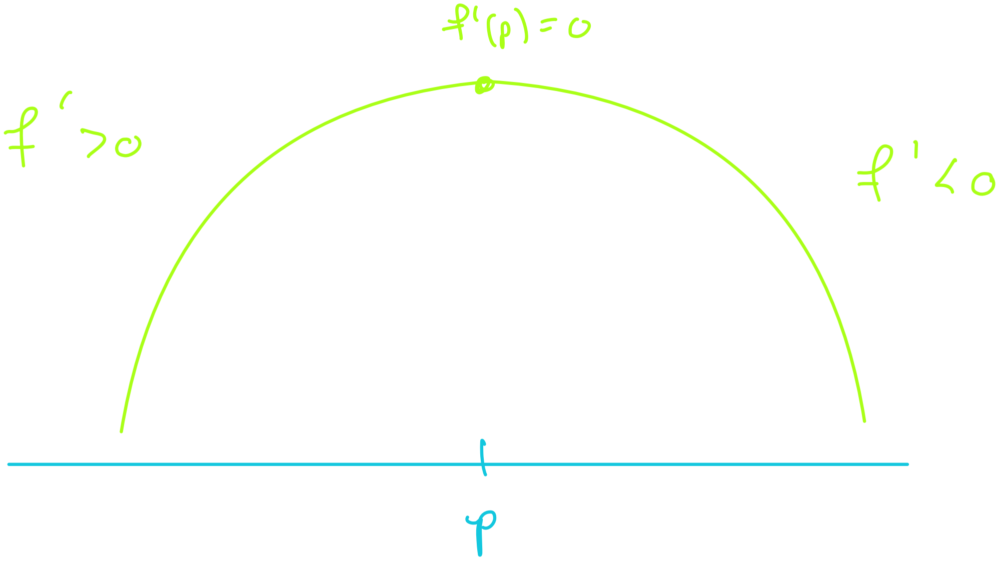
- If f' changes from pos. to neg. at p as we move left to right, then f has a local max. at p.

Ex:



(local min. case)

$f \searrow$ on $(-\infty, p)$ $\quad$ $f \nearrow$ on (p, ∞)



(local max. case)

$f \nearrow$ on $(-\infty, p)$ $\quad$ $f \searrow$ on (p, ∞)

Thm: (2^{nd} der. test for extrema) Suppose p is a critical pt of a function f and $f'(p)=0$.

- $f''(p)>0 \implies f$ has loc. min. at p
- $f''(p)<0 \implies f$ has loc. max. at p
- $f''(p)=0$ tells us nothing.

Ex: Use the 2^{nd} der. test on $f(x) = x^3-9x^2+21$ at $x=-2$ to test if this is a local min. or max.

$$\frac{d^2}{dx^2}\left(x^3-9x^2+21\right) = \frac{d}{dx}\left(\frac{d}{dx}\left(x^3-9x^2+21\right)\right)$$

$$= \frac{d}{dx}\left(\frac{d}{dx}(x^3) + \frac{d}{dx}(-9x^2) + \overset{0}{\cancel{\frac{d}{dx}(21)}}\right)$$

$$= \frac{d}{dx}\left(3x^2-18x\right) = 6x-18$$

$$f''(-2) = 6(-2)-18 = -12-18 = -36 < 0$$

$$f''(-2) < 0 \implies -2 \text{ is loc. max.}$$

[If 2^{nd} der. test doesn't help, then try 1^{st} der. test]

§4.2: Inflection pts

Obj: We study the pts on a graph where slope changes sign.

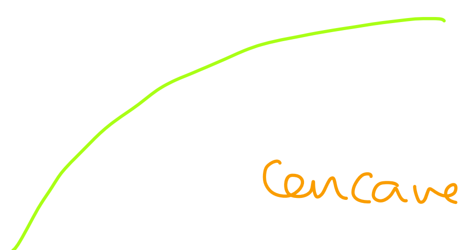
Def: A pt. on the graph of a function which changes concavity is called an "inflection pt."

Rmk: Recall the following:

Concave up, dec.

Concave up, inc.

concave down, dec.



Concave down, inc.

Rmk: Let f be a function. Suppose f'' is defined at both sides of a pt. p.

- If f'' is zero or undefined at p, then p is a potential inflection pt.

- To test if p is an inflection pt, check whether f'' changes sign at p.

Ex: $f(x) = x^3 - 9x^2 + 48x + 501$ at $x = 3$

$$f''(x) = \frac{d^2}{dx^2}(x^3 - 9x^2 + 48x + 501) = \frac{d}{dx}(3x^2 - 18x + 48)$$
$$= 6x - 18$$

$f''(3) = 6\cdot 3 - 18 = 0 \implies 3$ might be an infl. pt.

$$\left.\begin{array}{l} f''(2) = 6\cdot 2 - 18 = -6 \\ f''(4) = 6\cdot 4 - 18 = 6 \end{array}\right\} \implies f'' \text{ changes sign at } 3 \implies 3 \text{ is infl. pt.}$$

Ex: $f(x) = x^3 + 2x + 1$ at $x = 0$

$$f''(x) = \frac{d^2}{dx^2}(x^3 + 2x + 1) = \frac{d}{dx}(3x^2 + 2) = 6x$$

$f''(0) = 6\cdot 0 = 0$

$$\left.\begin{array}{l} f''(-1) = 6\cdot(-1) = -6 \\ f''(1) = 6\cdot 1 = 6 \end{array}\right\} \implies f'' \text{ changes sign at } 0 \implies 0 \text{ infl. pt.}$$

§4.3: Global maxima and minima

Obj: We study maxima and minima for the entire domain of a function.

Def: Let f be a function and $p \in \mathrm{dom}(f)$.

- p is a "global max." if for all $q \in \mathrm{dom}(f)$, one has $f(p) \geq f(q)$

- p is a "global min." if for all $q \in \mathrm{dom}(f)$, one has $f(q) \geq f(p)$.

Ex: $f(x) = x^2+1$ at $x=0$

Ex: $f(x) = x^3+1$ on $[0,1]$

Rmk: Let $f(x)$ be a function on $[a,b]$.
To find global max/min of f:

- Make a list of all $x \in [a,b]$ such that

  1. $f'(c) = 0$,

  2. $f'(c)$ doesn't exist, or

  3. $c = a$ or $c = b$

- Evaluate $f(c)$ amongst the list. Order from least to greatest. This tells you where gl. min/max's occur.

Ex: $f(x) = x^3 + 2x + 1$ on $[-1, 1]$